MATHEMATICAL METHODS IN PRICING RAINBOW OPTIONS

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A Thesis in Mathematics

Presented to the Faculties of the University of Pennsylvania
In Partial Fulfillment of the Requirements
For the Degree of Master of Arts

2003

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Supervisor of Thesis

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Graduate Group Chairman

Dedicatio

Haec thesis dedicata est ad honorem gloriamque
Sanctae Dei Genetricis Mariae,
Coredemptricis
Gratiarum Omnium Mediatricis
Advocatae
Solae Profligatricis Cunctarum Haeresum
Reginae Caeli et Terrae
Reginae Sanctae Sacrificiae Missae
Virginis Virginum
Immaculatae Conceptionis
Foederis Arcae
Verae Turris Eburneae

Abstract

In this paper, we present a new method for valuing options on two risky assets. It essentially allows these options to be looked from a more convenient perspective: as options on one risky asset whose price process may be a bit more complex than that for either of the assets. The method does have limitations and cannot be used for all possible options on two risky assets. However, its convenience and, more importantly, the ease with which one can know when it is possible to apply it make it a worthwhile tool for one to know.

I. Introduction

Suppose the assumptions of Black and Scholes hold and further suppose we have two assets whose prices follow geometric Brownian motion:

$$dS_t^i = S_t^i \mu_i dt + S_t^i \sigma_i dW_t^i$$

$$i = 1,2$$

$$Corr(W_t^1, W_t^2) = \rho$$

$$Cov = \rho\sigma_1\sigma_2$$

Further dW^i is a Wiener process. Consider the European option which can be exercised at time T and whose payoff is:

$$V_T(S_T^1, S_T^2) = \max(S_T^1 - kS_T^2, 0)$$

By the No Arbitrage Hypothesis, at all times t before expiry, we must have:

$$0 \le V_t(S_t^1, S_t^2) \le S_t^1$$

Effectively, this option gives its holder the right to exchange k shares of S^2 for one share of S^1.

II. Pricing the Option

To value this option, we will use the method of changing the numeraire. All of the above prices are dollar-valued prices. To price the option, we first determine its price in another currency, not dollars but shares of S^2 itself! Then the price of the option in dollars will simply be the price of the option in shares of S^2 multiplied by the dollar-price of S^2. We introduce the following notation change:

$A_t^{i,j}$ the price at time t of asset A^i valued in currency j

In S^2-*land*, where the currency is S^2, the price of S^2 is always 1. That is, rather than using dollar bills, the citizens of S^2-*land* use S^2 stock certificates. As stated before, this option gives its holder the right to exchange k shares of S^2 for one share of S^1. This will only be done if the price of S^1 in S^2-*land* is greater than k. Hence the final value of the option in S^2-*land* is:

$$V_T^{S^2}(S_T^{1,\$}, S_T^{2,\$}) = \max\left(S_T^{1,\$} \big/ S_T^{2,\$} - k, 0 \right)$$

or likewise

$$V_T^{S^2}(S_T^{1,S^2}) = \max\left(S_T^{1,S^2} - k, 0 \right)$$

since

$$S_t^{1,S^2} = S_t^{1,\$} \big/ S_t^{2,\$}$$

Now, to price this option we must look at the differential equations satisfied by the assets. First and foremost:

$$\begin{aligned}
dS_t^{1,S^2} &= \frac{\partial S_t^{1,S^2}}{\partial S_t^{1,\$}} dS_t^{1,\$} + \frac{1}{2} \frac{\partial^2 S_t^{1,S^2}}{\partial S_t^{1,\$ 2}} (dS_t^{1,\$})^2 \\
&\quad + \frac{\partial S_t^{1,S^2}}{\partial S_t^{2,\$}} dS_t^{2,\$} + \frac{1}{2} \frac{\partial^2 S_t^{1,S^2}}{\partial S_t^{2,\$ 2}} (dS_t^{3,\$})^2 + \frac{\partial^2 S_t^{1,S^2}}{\partial S_t^{1,\$} \partial S_t^{2,\$}} (dS_t^{1,\$})(dS_t^{2,\$}) \\
&= \frac{1}{S_t^{2,\$}} \left[S_t^{1,\$} \mu_1 dt + S_t^{1,\$} \sigma_1 dW_t^1 \right] + 0 + \frac{-S_t^{1,\$}}{S_t^{2,\$ 2}} \left[S_t^{2,\$} \mu_2 dt + S_t^{2,\$} \sigma_2 dW_t^2 \right] \\
&\quad + \frac{S_t^{1,\$}}{S_t^{2,\$ 3}} \left[S_t^{2,\$ 2} \sigma_2{}^2 dt \right] + \frac{-1}{S_t^{2,\$ 2}} \left[S_t^{1,\$} \sigma_1 S_t^{2,\$} \sigma_2 dW_t^1 dW_t^2 \right] \\
&= \frac{S_t^{1,\$}}{S_t^{2,\$}} \left[\mu_1 - \mu_2 + \sigma_2{}^2 - Cov \right] dt + \frac{S_t^{1,\$}}{S_t^{2,\$}} \sigma_1 dW_t^1 - \frac{S_t^{1,\$}}{S_t^{2,\$}} \sigma_2 dW_t^2 \\
&= S_t^{1,S^2} \left[(\mu_1 - \mu_2 + \sigma_2{}^2 - Cov) dt + \sigma_1 dW_t^1 + \sigma_2 dW_t^2 \right]
\end{aligned}$$

See Appendix 1 for proof that:

$$dW_t^1 dW_t^2 = \rho dt$$

From the above it is clear that,

$$\left(dS_t^{1,S^2}\right)^2 = S_t^{1,S^2\,2}\left[\sigma_1^{\,2} + \sigma_2^{\,2} - 2Cov\right]dt$$

Now, at all times, the option V^{S^2}, is a function of two variables, S^{1,S^2} and t. Hence:

$$dV_t^{S^2} = \frac{\partial V^{S^2}}{\partial t}dt + \frac{\partial V^{S^2}}{\partial S^{1,S^2}}dS^{1,S^2} + \frac{1}{2}\frac{\partial^2 V^{S^2}}{\partial S^{1,S^2\,2}}\left(dS^{1,S^2}\right)^2$$

$$= \left[\frac{\partial V^{S^2}}{\partial t} + \frac{\partial V^{S^2}}{\partial S^{1,S^2}}S^{1,S^2}(M) + \frac{1}{2}\frac{\partial^2 V^{S^2}}{\partial S^{1,S^2\,2}}S^{1,S^2\,2}(\Sigma^2)\right]dt$$
$$+ \frac{\partial V^{S^2}}{\partial S^{1,S^2}}S^{1,S^2}\sigma_1 dW_t^1 - \frac{\partial V^{S^2}}{\partial S^{1,S^2}}S^{1,S^2}\sigma_2 dW_t^2$$

$$M = \mu_1 - \mu_2 + \sigma_2^{\,2} - Cov$$

$$\Sigma^2 = \sigma_1^{\,2} + \sigma_2^{\,2} - 2Cov$$

In S^2-*land*, we shall form a self-financing replicating portfolio which will be composed of two assets: the stock (that is, S^1) and the risk-free asset (that is, S^2 whose risk-free rate of return is its dividend rate):

$$\Pi_t^{S^2} = \varphi_t S_t^{1,S^2} + \psi_t S_t^{2,S^2}$$

$$d\Pi_t^{S^2} = \varphi_t dS_t^{1,S^2} + \psi_t dS_t^{2,S^2}$$

Simplifying:

$$d\Pi_t^{S^2} = \left[\varphi_t S_t^{1,S^2}(\mu_1 - \mu_2 + \sigma_1^{\,2} - Cov) + \psi_t r S_t^{2,S^2}\right]dt + \varphi_t S_t^{1,S^2}\sigma_1 dW_t^1 - \varphi_t S_t^{1,S^2}\sigma_2 dW_t^2$$

We also know that:

$$dS_t^{2,S^2} = rS^{2,S^2}dt$$

where r is the dividend that S^2 pays out valued in S^2-*land* [i]. Note, if S^2 does not pay a dividend, $r = 0$ and we can replace S_t^{2,S^2} by 1. Now, we must adjust φ_t and ψ_t such that:

$$\Pi_t^{S^2} = V_t^{S^2}$$

$$d\Pi_t^{S^2} = dV_t^{S^2}$$

We set

$$\varphi_t = \frac{\partial V^{S^2}}{\partial S^{1,S^2}}$$

This cancels the two Brownian motion terms in the equation for dV^{S^2} . Now we must solve:

$$\frac{\partial V^{S^2}}{\partial t} + \frac{\partial V^{S^2}}{\partial S^{1,S^2}} S^{1,S^2}(\mu_1 - \mu_2 + \sigma_2{}^2 - Cov) + \frac{1}{2}\frac{\partial^2 V^{S^2}}{\partial S^{1,S^2\,2}} S^{1,S^2\,2}(\sigma_1{}^2 + \sigma_2{}^2 - 2Cov)$$

$$= \varphi_t S_t^{1,S^2}(\mu_1 - \mu_2 + \sigma_2{}^2 - Cov) + \psi_t r S_t^{2,S^2}$$

Using

$$\varphi_t = \frac{\partial V^{S^2}}{\partial S^{1,S^2}}$$

$$\psi_t S_t^{2,S^2} = \left[\Pi_t^{S'} - \frac{\partial V^{S^2}}{\partial S^{1,S^2}} S^{1,S^2}\right] = \left[V_t^{S^2} - \frac{\partial V^{S^2}}{\partial S^{1,S^2}} S^{1,S^2}\right]$$

We can simplify the equation to:

$$\frac{\partial V^{S^2}}{\partial t} + rS_t^{1,S^2}\frac{\partial V^{S^2}}{\partial S^{1,S^2}} + \frac{1}{2} S_t^{1,S^2\,2}(\sigma_1{}^2 + \sigma_2{}^2 - 2Cov)\frac{\partial^2 V^{S^2}}{\partial S^{1,S^2\,2}} - rV_t^{S^2} = 0$$

This is of exactly the same form as the original Black Scholes equation! Hence, we know that[ii]:

$$V_t^{S^2} = S_t^{1,S^2} N(d_1) - ke^{-r(T-t)} N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_t^{1,S^2}}{k}\right) + \left(r + \frac{1}{2}\Sigma^2\right)(T-t)}{\Sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{S_t^{1,S^2}}{k}\right) + \left(r - \frac{1}{2}\Sigma^2\right)(T-t)}{\Sigma\sqrt{T-t}}$$

$$\Sigma^2 = \sigma_1{}^2 + \sigma_2{}^2 - 2Cov$$

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{\left(-\frac{1}{2}\right)^2} ds$$

Now, this gives the value of the option in S^2-*land*. To value it in dollars, we need to convert using the dollar-to-S^2 exchange rate (that is, the dollar price of S^2 shares):

$$V_t^{\$} = S_t^{2,\$} V_t^{S^2}$$

Valuing everything in terms of dollars, we see that this means:

$$V_t^{\$} = S_t^{1,\$} N(d_1) - kS_t^{2,\$} e^{-r(T-t)} N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_t^{1,\$}}{kS_t^{2,\$}}\right) + \left(r + \frac{1}{2}\Sigma^2\right)(T-t)}{\Sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{S_t^{1,\$}}{kS_t^{2,\$}}\right) + \left(r - \frac{1}{2}\Sigma^2\right)(T-t)}{\Sigma\sqrt{T-t}}$$

$$\Sigma^2 = \sigma_1{}^2 + \sigma_2{}^2 - 2Cov$$

III. The American Option

We now consider the American option analogous to the option above. That is, consider an option which can be exercised at time t less than or equal to T and whose payoff is:

$$A_t^{\$}(S_t^{1,\$}, S_t^{2,\$}) = \max(S_t^{1,\$} - kS_t^{2,\$}, 0)$$

It is easy to show that the American option has the same value as the European option. Consider the following two portfolios:

Portfolio I: Long one European option

Portfolio II: Long one share of $S^{1,\$}$; short two shares of $S^{2,\$}$

Now, at any time t, the portfolios will have the value:

Portfolio I: $V_t^\$(S_t^{1,\$}, S_t^{2,\$})$

Portfolio II: $S_t^{1,\$} - kS_t^{2,\$}$

At expiration, the portfolios will have the value:

Portfolio I: $\max(S_T^{1,\$} - kS_T^{2,\$}, 0)$

Portfolio II: $S_T^{1,\$} - kS_T^{2,\$}$

Since Portfolio I is always worth as much as or more than Portfolio II, we must have:

$$V_t^\$(S_t^{1,\$}, S_t^{2,\$}) \geq S_t^{1,\$} - kS_t^{2,\$}$$

Hence, the value of the European option exceeds what one would get in return for exercising an American option early. Hence, one would never exercise an American option early and its value will be identical to the European option:

$$A_t^\$(S_t^{1,\$}, S_t^{2,\$}) = V_t^\$(S_t^{1,\$}, S_t^{2,\$})$$

IV. An Alternative Derivation

There is an alternative derivation for this option presented by Margrabe. It is briefly sketched in this section. The option at some time prior to expiry is a function of the time and the dollar prices of S^1 and S^2. That is:

$$V = V_t^\$(S_t^{1,\$}, S_t^{2,\$})$$

For a fixed time t in a perfect market:

$$\lambda V_t^\$(S_t^{1,\$}, S_t^{2,\$}) = V_t^\$(\lambda S_t^{1,\$}, \lambda S_t^{2,\$})$$

That is, λ of these options that allow the holder to exchange k shares of $S^{2,\$}$ for $S^{1,\$}$ should have the same value as one option that allows the holder to exchange λk shares of $S^{2,\$}$ for λ shares of $S^{1,\$}$. Since this is the case, we can use Euler's Theorem:

$$V_t^{\$}(S_t^{1,\$}, S_t^{2,\$}) = \frac{\partial V_t^{\$}}{\partial S_t^{1,\$}} S_t^{1,\$} + \frac{\partial V_t^{\$}}{\partial S_t^{2,\$}} S_t^{2,\$}$$

$$\Rightarrow V_t^{\$}(S_t^{1,\$}, S_t^{2,\$}) - \frac{\partial V_t^{\$}}{\partial S_t^{1,\$}} S_t^{1,\$} - \frac{\partial V_t^{\$}}{\partial S_t^{2,\$}} S_t^{2,\$} = 0$$

This could be thought of as a portfolio that is long one unit option and short the appropriate partial derivative units of each stock. Now, over any short time horizon we see that any return is zero:

$$dV_t^{\$}(S_t^{1,\$}, S_t^{2,\$}) - \frac{\partial V_t^{\$}}{\partial S_t^{1,\$}} dS_t^{1,\$} - \frac{\partial V_t^{\$}}{\partial S_t^{2,\$}} dS_t^{2,\$} = 0$$

But, by the application of Ito's lemma to $V_t^{\$}$ we see that:

$$\begin{aligned} dV_t^{\$} &= \frac{\partial V_t^{\$}}{\partial t} dt + \frac{\partial V_t^{\$}}{\partial S_t^{1,\$}} dS_t^{1,\$} + \frac{\partial V_t^{\$}}{\partial S_t^{2,\$}} dS_t^{2,\$} \\ &= \left[\frac{1}{2} \frac{\partial^2 V_t^{\$}}{\partial S_t^{1,\$\,2}} \sigma_1{}^2 S_t^{1,\$\,2} + \frac{\partial^2 V_t^{\$}}{\partial S_t^{1,\$} \partial S_t^{2,\$}} S_t^{1,\$} S_t^{2,\$} Cov + \frac{1}{2} \frac{\partial^2 V_t^{\$}}{\partial S_t^{2,\$\,2}} \sigma_2{}^2 S_t^{2,\$\,2} \right] dt \end{aligned}$$

By comparing the two equations, we see that:

$$\frac{\partial V_t^{\$}}{\partial t} + \frac{1}{2} \frac{\partial^2 V_t^{\$}}{\partial S_t^{1,\$\,2}} \sigma_1{}^2 S_t^{1,\$\,2} + \frac{\partial^2 V_t^{\$}}{\partial S_t^{1,\$} \partial S_t^{2,\$}} S_t^{1,\$} S_t^{2,\$} Cov + \frac{1}{2} \frac{\partial^2 V_t^{\$}}{\partial S_t^{2,\$\,2}} \sigma_2{}^2 S_t^{2,\$\,2} = 0$$

According to Margrabe, the solution to the above differential equation is the equation presented in Section II of this paper. In other words, it is the *$-valued* equivalent of the differential equation derived and solved in the previous section.

V. Extensions

Again, suppose we have two assets whose prices follow geometric Brownian motion:

$$dS_t^{i,\$} = S_t^{i,\$} \mu_i dt + S_t^{i,\$} \sigma_i dW_t^i$$

$$i = 1,2$$

$$Corr(W_t^1, W_t^2) = \rho$$

$$Cov = \rho\sigma_1\sigma_2$$

Consider the European option which can be exercised at time *T* and whose payoff is:

$$V_T^{\$}(S_T^{1,\$}, S_T^{2,\$}) = \max(S_T^{1,\$}, S_T^{2,\$})$$

Using the method discussed previously, we know that in *S^2-land*, this option has payoff:

$$V_T^{S^2}(S_T^{1,S^2}) = \max(S_T^{1,S^2}, 1) = \max\left(S_T^{1,\$} \big/ S_T^{2,\$}, 1 \right)$$

Now, the equations of Section III still apply:

$$dS_t^{1,S^2} = S_t^{1,S^2}\left[(\mu_1 - \mu_2 + \sigma_2{}^2 - Cov)dt + \sigma_1 dW_t^1 + \sigma_2 dW_t^2 \right]$$

$$\begin{aligned} dV_t^{S^2} &= \frac{\partial V^{S^2}}{\partial t}dt + \frac{\partial V^{S^2}}{\partial S^{1,S^2}}dS^{1,S^2} + \frac{1}{2}\frac{\partial^2 V^{S^2}}{\partial S^{1,S^2\,2}}\left(dS^{1,S^2}\right)^2 \\ &= \left[\frac{\partial V^{S^2}}{\partial t} + \frac{\partial V^{S^2}}{\partial S^{1,S^2}} S^{1,S^2}(M) + \frac{1}{2}\frac{\partial^2 V^{S^2}}{\partial S^{1,S^2\,2}} S^{1,S^2\,2}(\Sigma^2) \right] dt \\ &\quad + \frac{\partial V^{S^2}}{\partial S^{1,S^2}} S^{1,S^2}\sigma_1 dW_t^1 - \frac{\partial V^{S^2}}{\partial S^{1,S^2}} S^{1,S^2}\sigma_2 dW_t^2 \end{aligned}$$

$$M = \mu_1 - \mu_2 + \sigma_2{}^2 - Cov$$

$$\Sigma^2 = \sigma_1{}^2 + \sigma_2{}^2 - 2Cov$$

So, we can use the same replicating strategy to derive the same differential equation:

$$\frac{\partial V^{S^2}}{\partial t} + rS_t^{1,S^2}\frac{\partial V^{S^2}}{\partial S^{1,S^2}} + \frac{1}{2}S_t^{1,S^2\,2}(\sigma_1{}^2 + \sigma_2{}^2 - 2Cov)\frac{\partial^2 V^{S^2}}{\partial S^{1,S^2\,2}} - rV_t^{S^2} = 0$$

Thus, this option is identical to the previous on in everything respect except the boundary condition. The solution to the differential equation with the new boundary condition is presented in Appendix III. It is:

$$V_t^{S^2} = e^{-r(T-t)} + S_t^{1,S^2}N(d_1) - e^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_t^{1,\$}}{kS_t^{2,\$}}\right) + \left(r + \frac{1}{2}\Sigma^2\right)(T-t)}{\Sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{S_t^{1,\$}}{kS_t^{2,\$}}\right) + \left(r - \frac{1}{2}\Sigma^2\right)(T-t)}{\Sigma\sqrt{T-t}}$$

$$\Sigma^2 = \sigma_1{}^2 + \sigma_2{}^2 - 2Cov$$

Valued in dollars, this is:

$$V_t^{\$} = S_t^{2,\$}e^{-r(T-t)} + S_t^{1,\$}N(d_1) - S_t^{2,\$}e^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_t^{1,\$}}{kS_t^{2,\$}}\right) + \left(r + \frac{1}{2}\Sigma^2\right)(T-t)}{\Sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{S_t^{1,\$}}{kS_t^{2,\$}}\right) + \left(r - \frac{1}{2}\Sigma^2\right)(T-t)}{\Sigma\sqrt{T-t}}$$

$$\Sigma^2 = \sigma_1{}^2 + \sigma_2{}^2 - 2Cov$$

This formula looks suspiciously familiar. In fact, it should be clear that:

$$V_t^{\$} = S_t^{2,\$}e^{-r(T-t)} + O_t^{\$}$$

where $O^{\$}$ is the option of Section II with $k = 1$. To see why this is so, consider the following two portfolios[iii]:

Portfolio I: Long one share $S^{2,\$}$; long one option with payoff

$\max(S_T^{1,\$} - S_T^{2,\$}, 0)$

Portfolio II: Long one option with payoff $\max(S_T^{1,\$}, S_T^{2,\$})$

If $S^{1,\$}$ is greater than $S^{2,\$}$ at expiration, then the portfolios will be worth:

Portfolio I: $S^{2,\$} + (S^{1,\$} - S^{2,\$}) = S^{1,\$}$

Portfolio II: $S^{1,\$}$

If $S^{1,\$}$ is less than or equal to than $S^{2,\$}$ at expiration, then the portfolios will be worth:

Portfolio I: $S^{2,\$} + (0) = S^{2,\$}$

Portfolio II: $S^{2,\$}$

Since Portfolio I is worth the same amount as Portfolio II at expiration in all cases, the two portfolios must always be worth the same amount by the No Arbitrage Hypothesis. Thus, we get the result obtained above with $r = 0$:

$$V_t^{\$} = S_t^{2,\$} + O_t^{\$}$$

Now we consider a related option with payoff:

$$V_T^{\$}(S_T^{1,\$}, S_T^{2,\$}) = \min(S_T^{1,\$}, S_T^{2,\$})$$

While we can use a method similar to that in Appendix III to value this option, greater intuition can be derived by considering the No Arbitrage Argument presented below[iv]:

Portfolio I: Long one option which pays off $\min(S_T^{1,\$}, S_T^{2,\$})$

Portfolio II: Long one share $S^{1,\$}$; short one option with payoff $\max(S_T^{1,\$} - S_T^{2,\$}, 0)$

If $S^{1,\$}$ is greater than $S^{2,\$}$ at expiration, then the portfolios will be worth:

Portfolio I: $S^{2,\$}$

Portfolio II: $S^{1,\$} - (S^{1,\$} - S^{2,\$}) = S^{2,\$}$

If $S^{1,\$}$ is less than or equal to than $S^{2,\$}$ at expiration, then the portfolios will be worth:

Portfolio I: $S^{1,\$}$

Portfolio II: $S^{1,\$} + (0) = S^{1,\$}$

Since Portfolio I is worth the same amount as Portfolio II at expiration in all cases, the two portfolios must always be worth the same amount by the No Arbitrage Hypothesis. Thus, we get the following result:

$$V_t^{\$} = S_t^{1,\$} - O_t^{\$}$$

where $O_t^{\$}$ is the option which pays off $\max(S_T^{1,\$} - S_T^{2,\$}, 0)$. Hence, the value of this option is:

$$V_t^{\$} = S_t^{1,\$} - S_t^{1,\$} N(d_1) + S_t^{2,\$} e^{-r(T-t)} N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_t^{1,\$}}{kS_t^{2,\$}}\right) + \left(r + \frac{1}{2}\Sigma^2\right)(T-t)}{\Sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{S_t^{1,\$}}{kS_t^{2,\$}}\right) + \left(r - \frac{1}{2}\Sigma^2\right)(T-t)}{\Sigma\sqrt{T-t}}$$

$$\Sigma^2 = \sigma_1{}^2 + \sigma_2{}^2 - 2Cov$$

Appendix 1

Proposition: Suppose B_t^1 and B_t^2 are two Brownian motions. Then, $dB_t^1 dB_t^2 = \rho_{1,2} dt$.

Proof: We shall show this in two stages. First, we shall show that for independent Brownian motions, $dB_t^1 dB_t^2 = 0$. Then, using this result, we shall show the more general formula.

Part I:

Let $\wp = \{t_0, ..., t_n\}$ be a partition of [0,T]. Then, define:

$$S_\wp = \sum_{k=0}^{n-1} (B_{t_{k+1}}^1 - B_{t_k}^1)(B_{t_{k+1}}^2 - B_{t_k}^2)$$

Now, all the increments above are independent of each other and have mean zero. Hence:

$$E(S_\wp) = 0$$

Now we compute the variance of $S_\wp$:

$$Var(S_\wp) = E(S_\wp{}^2) - E(S_\wp)^2 = E(S_\wp{}^2)$$

$$\begin{aligned} E(S_\wp{}^2) \quad = \quad & E\Bigg(\sum_{k=0}^{n-1} (B_{t_{k+1}}^1 - B_{t_k}^1)^2 (B_{t_{k+1}}^2 - B_{t_k}^2)^2 \\ & + 2\sum_{j<k}^{n-1} (B_{t_{j+1}}^1 - B_{t_j}^1)(B_{t_{k+1}}^1 - B_{t_k}^1)(B_{t_{j+1}}^2 - B_{t_j}^2)(B_{t_{k+1}}^2 - B_{t_k}^2) \Bigg) \end{aligned}$$

In the second sum, all of the increments are independent and have mean zero. Hence:

$$Var(S_\wp) = E\left(\sum_{k=0}^{n-1} (B_{t_{k+1}}^1 - B_{t_k}^1)^2 (B_{t_{k+1}}^2 - B_{t_k}^2)^2 \right)$$

Now, $(B_{t_{k+1}}^1 - B_{t_k}^1)^2$ and $(B_{t_{k+1}}^2 - B_{t_k}^2)^2$ are independent of each other. Moreover, each has expectation $(t_{k+1} - t_k)$. Thus:

$$Var(S_{\wp}) = \sum_{k=0}^{n-1} (t_{k+1} - t_k)^2 \leq \|\wp\| \sum_{k=0}^{n-1} (t_{k+1} - t_k) = \|\wp\| T$$

As $\|\wp\| \to 0$, we have $Var(S_{\wp}) \to 0$ so $S_{\wp}$ converges to the constant $E(S_{\wp}) = 0$.

Part II:

First we shall show that two normal distributions are independent if and only if their covariance is zero. We will use this in conjunction with the above to prove the rest of the theorem. Suppose X and Y are two independent normal distributions. Then Cov(X,Y) = E[(X-μ_X)(Y-μ_Y)] = E[XY] - $\mu_X\mu_Y$ = E[X]E[Y] - $\mu_X\mu_Y$ = $\mu_X\mu_Y$ - $\mu_X\mu_Y$ = 0. Now, suppose X and Y are two correlated normal distributions with correlation coefficient ρ = Cov(X,Y)/$\sigma_X\sigma_Y$. This means that X and Y come from a bivariate normal distribution:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2}\left(\frac{1}{1-\rho^2}\right)\left(\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho\frac{(x-\mu_X)}{\sigma_X}\frac{(y-\mu_Y)}{\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)}$$

If ρ = 0, then

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_X\sigma_Y}\exp\left\{-\frac{1}{2}\left(\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)\right\} \\ &= \left[\frac{1}{\sqrt{2\pi}\sigma_X}\exp\left\{-\frac{1}{2}\frac{(x-\mu_X)^2}{\sigma_X^2}\right\}\right]\left[\frac{1}{\sqrt{2\pi}\sigma_Y}\exp\left\{-\frac{1}{2}\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right\}\right] \\ &= f_X(x) f_Y(y) \end{aligned}$$

Hence, X and Y are independent.

Now, suppose B_t^1 and B_t^2 are two Brownian motions. Then, $B_t^i \sim N(0,t)$. That is, they are distributed according to the normal distribution with mean *0* and variance *t*. Now, suppose $Cov(B_t^1, B_t^2) = Ct$. Then, $\rho_{1,2} = Ct / \sqrt{t}\sqrt{t} = C$. Now, define the following random variables:

$$U_t^1 = B_t^1$$

$$U_t^2 = \frac{-C}{\sqrt{1-C^2}} B_t^1 + \frac{1}{\sqrt{1-C^2}} B_t^2$$

Then, $U_t^1 \sim N(0,t)$ and $U_t^2 \sim N\left(0, \left({}^{1+C^2}\!/_{1-C^2}\right)\right)$. Furthermore:

$$\begin{aligned} Cov(U_t^1, U_t^2) &= E(U_t^1 U_t^2) \\ &= \frac{-C}{\sqrt{1-C^2}} E(B_t^1 B_t^1) + \frac{1}{\sqrt{1-C^2}} E(B_t^1 B_t^2) \\ &= \frac{-C}{\sqrt{1-C^2}} t + \frac{1}{\sqrt{1-C^2}} Ct \\ &= 0 \end{aligned}$$

Hence, the U^i are uncorrelated and independent. Now, it is clear that:

$$dU_t^1 = dB_t^1$$

$$dU_t^2 = \frac{-C}{\sqrt{1-C^2}} dB_t^1 + \frac{1}{\sqrt{1-C^2}} dB_t^2$$

However, since by the first part of this proof $dU_t^1 dU_t^2 = 0$, we know that:

$$dU_t^1 dU_t^2 = 0 = \frac{-C}{\sqrt{1-C^2}} \left(dB_t^1\right)^2 + \frac{1}{\sqrt{1-C^2}} dB_t^1 dB_t^2 = \frac{-C}{\sqrt{1-C^2}} dt + \frac{1}{\sqrt{1-C^2}} dB_t^1 dB_t^2$$

$$\Rightarrow \frac{C}{\sqrt{1-C^2}} dt = \frac{1}{\sqrt{1-C^2}} dB_t^1 dB_t^2$$

$$\Rightarrow Cdt = dB_t^1 dB_t^2$$

That is $dB_t^1 dB_t^2 = \rho dt$ where ρ is the "instantaneous" correlation between the two Brownian motions.

Appendix II

Proposition: The differential equation:

$$\frac{\partial C}{\partial t} + bS^2 \frac{\partial^2 C}{\partial S^2} + cS\frac{\partial C}{\partial S} + dC = 0$$

$$C(0,t) = 0$$

$$C(S,t) \sim S \text{ as } S \to \infty$$

$$C(S,T) = \max(S - K, 0)$$

has solution:

$$C = Se^{(c+d)(T-t)}N(d_+) - Ke^{d(T-t)}N(d_-)$$

$$d_+ = \frac{\ln\left(\frac{S}{K}\right) + (c+b)(T-t)}{\sqrt{2b(T-t)}}$$

$$d_- = \frac{\ln\left(\frac{S}{K}\right) + (c-b)(T-t)}{\sqrt{2b(T-t)}}$$

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{\left(-\frac{1}{2}\right)^2}\, ds$$

Proof: Consider the following three substitutions:

- $\tau = b(T-t)$ which implies that $t = T - \tau/b$, $\frac{d\tau}{dt} = -b$ and $\tau = 0 \Leftrightarrow t = T$
- $S = Ke^x$ which implies that $x = \ln\left(S/K\right)$
- $C(S,t) = Ku(x,\tau)$

The final substitution in tandem with the fact that:

$$C(S,T) = \max(S-K,0) = \max(Ke^x - K, 0) = K\max(e^x - 1, 0)$$

implies that $u(x,0) = \max(e^x - 1, 0)$.

The above substitutions mean:

- $\frac{\partial C}{\partial t} = \frac{\partial C}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{\partial (Ku)}{\partial \tau}(-b) = -Kb\frac{\partial u}{\partial \tau}$

- $\frac{\partial C}{\partial S} = \frac{\partial C}{\partial x}\frac{\partial x}{\partial S} = \frac{\partial (Ku)}{\partial x}\left(\frac{1}{S}\right) = \frac{K}{S}\frac{\partial u}{\partial x}$

- $$\begin{aligned}\frac{\partial^2 C}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{K}{S}\frac{\partial u}{\partial x}\right) = \frac{-K}{S^2}\frac{\partial u}{\partial x} + \frac{K}{S}\frac{\partial}{\partial S}\frac{\partial u}{\partial x} = \frac{-K}{S^2}\frac{\partial u}{\partial x} + \frac{K}{S}\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial S} \\ &= \frac{-K}{S^2}\frac{\partial u}{\partial x} + \frac{K}{S^2}\frac{\partial^2 u}{\partial x^2}\end{aligned}$$

Plugging this back into the original differential equation and solving for $\frac{\partial u}{\partial \tau}$ yields the following:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \left(\frac{c}{b} - 1\right)\frac{\partial u}{\partial x} + \frac{d}{b}u$$

Now, we set:

$$\alpha = -\frac{1}{2}\left(\frac{c}{b} - 1\right)$$

$$\beta = \frac{d}{b} - \frac{1}{4}\left(\frac{c}{b} - 1\right)^2$$

Then

$$u(x,\tau) = e^{\alpha x + \beta \tau}v(x,\tau) = e^{-\frac{1}{2}\left(\frac{c}{b}-1\right)x + \left(\frac{d}{b} - \frac{1}{4}\left(\frac{c}{b}-1\right)^2\right)\tau}v(x,\tau)$$

where *v* is a function satisfying

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2}$$

Therefore,

$$u(x,0) = \max\left(e^x - 1, 0\right) = e^{-\frac{1}{2}\left(\frac{c}{b}-1\right)x}v(x,0)$$

$$\Rightarrow v(x,0) = \max\left(e^{\frac{1}{2}\left(\frac{c}{b}+1\right)x} - e^{\frac{1}{2}\left(\frac{c}{b}-1\right)x}, 0 \right)$$

Now, we must solve the differential equation[v]:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2}$$

$$v(x,0) = \max\left(e^{\frac{1}{2}\left(\frac{c}{b}+1\right)x} - e^{\frac{1}{2}\left(\frac{c}{b}-1\right)x}, 0 \right)$$

This is the Heat Equation with initial condition $\varphi(x) = v(x,0)$. Its solution is:

$$v(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4\tau}} \varphi(y)dy.$$

Now, we make the change of variable $Q = \frac{y-x}{\sqrt{2\tau}}$. Our integral then becomes:

$$v(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-Q^2/2} \varphi(\sqrt{2\tau}Q + x)dQ$$

Now,

$$\varphi(y) \geq 0 \Leftrightarrow e^{\frac{1}{2}\left(\frac{c}{b}+1\right)y} \geq e^{\frac{1}{2}\left(\frac{c}{b}-1\right)y} \Leftrightarrow y \geq -y$$

Therefore,

$$\varphi(\sqrt{2\tau}Q + x) \geq 0 \Leftrightarrow \sqrt{2\tau}Q + x \geq -\sqrt{2\tau}Q - x \Leftrightarrow Q \geq \frac{-x}{\sqrt{2\tau}}$$

Thus, the integral becomes:

$$v(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-Q^2/2} \left[e^{\frac{1}{2}\left(\frac{c}{b}+1\right)\left(\sqrt{2\tau}Q+x\right)} - e^{\frac{1}{2}\left(\frac{c}{b}-1\right)\left(\sqrt{2\tau}Q+x\right)} \right] dQ = I_+ - I_-$$

$$I_+ = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{-Q^2}{2} + \frac{1}{2}\left(\frac{c}{b}+1\right)\left(\sqrt{2\tau}Q+x\right)} dQ$$

$$I_{-} = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{-Q^2}{2} + \frac{1}{2}\left(c/b - 1\right)\left(\sqrt{2\tau}Q + x\right)} dQ$$

We shall now calculate I_{+}:

$$\begin{aligned}
I_{+} &= \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{-Q^2}{2} + \frac{1}{2}\left(c/b + 1\right)\left(\sqrt{2\tau}Q + x\right)} dQ \\
&= \frac{e^{\frac{1}{2}\left(c/b + 1\right)x}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{-Q^2}{2} + \frac{1}{2}\left(c/b + 1\right)\sqrt{2\tau}Q} dQ \\
&= \frac{e^{\frac{1}{2}\left(c/b + 1\right)x}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\left(Q - \left(c/b + 1\right)\sqrt{2\tau}\right)^2 + \frac{1}{4}\left(c/b + 1\right)^2 \tau} dQ \\
&= \frac{e^{\frac{1}{2}\left(c/b + 1\right)x + \frac{1}{4}\left(c/b + 1\right)^2 \tau}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\left(Q - \left(c/b + 1\right)\sqrt{2\tau}\right)^2} dQ
\end{aligned}$$

Now, we make the substitution $\rho = Q - \frac{1}{2}\left(\frac{c}{b} + 1\right)\sqrt{2\tau}$. Since

$Q \geq \frac{-x}{\sqrt{2\tau}} \Leftrightarrow \rho \geq \frac{-x}{\sqrt{2\tau}} - \frac{1}{2}\left(\frac{c}{b} + 1\right)\sqrt{2\tau}$, the integral becomes:

$$I_{+} = \frac{e^{\frac{1}{2}\left(c/b + 1\right)x + \frac{1}{4}\left(c/b + 1\right)^2 \tau}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}} - \frac{1}{2}\left(\frac{c}{b} + 1\right)\sqrt{2\tau}}^{\infty} e^{-\frac{\rho^2}{2}} dQ = \frac{e^{\frac{1}{2}\left(c/b + 1\right)x + \frac{1}{4}\left(c/b + 1\right)^2 \tau}}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2\tau}} + \frac{1}{2}\left(\frac{c}{b} + 1\right)\sqrt{2\tau}} e^{-\frac{\rho^2}{2}} dQ$$

Written more simply, it is:

$$I_{+} = e^{\frac{1}{2}\left(c/b + 1\right)x + \frac{1}{4}\left(c/b + 1\right)^2 \tau} N(d_{+})$$

$$d_{+} = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}\left(\frac{c}{b} + 1\right)\sqrt{2\tau}$$

Likewise,

$$
\begin{aligned}
I_- &= \frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{-Q^2}{2}+\frac{1}{2}\left(c/b-1\right)\left(\sqrt{2\tau}Q+x\right)}dQ \\
&= \frac{e^{\frac{1}{2}\left(c/b-1\right)x}}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{-Q^2}{2}+\frac{1}{2}\left(c/b-1\right)\sqrt{2\tau}Q}dQ \\
&= \frac{e^{\frac{1}{2}\left(c/b-1\right)x}}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\left(Q-\left(c/b-1\right)\sqrt{2\tau}\right)^2+\frac{1}{4}\left(c/b-1\right)^2\tau}dQ \\
&= \frac{e^{\frac{1}{2}\left(c/b-1\right)x+\frac{1}{4}\left(c/b-1\right)^2\tau}}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\left(Q-\left(c/b-1\right)\sqrt{2\tau}\right)^2}dQ
\end{aligned}
$$

Now, we make the substitution $\gamma = Q-\frac{1}{2}\left(\frac{c}{b}-1\right)\sqrt{2\tau}$. Since

$Q \geq \frac{-x}{\sqrt{2\tau}} \Leftrightarrow \gamma \geq \frac{-x}{\sqrt{2\tau}}-\frac{1}{2}\left(\frac{c}{b}-1\right)\sqrt{2\tau}$, the integral becomes:

$$
I_- = \frac{e^{\frac{1}{2}\left(c/b-1\right)x+\frac{1}{4}\left(c/b-1\right)^2\tau}}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}-\frac{1}{2}\left(\frac{c}{b}-1\right)\sqrt{2\tau}}^{\infty} e^{-\frac{\gamma^2}{2}}dQ = \frac{e^{\frac{1}{2}\left(c/b-1\right)x+\frac{1}{4}\left(c/b-1\right)^2\tau}}{\sqrt{2\pi}}\int_{-\infty}^{\frac{x}{\sqrt{2\tau}}+\frac{1}{2}\left(\frac{c}{b}-1\right)\sqrt{2\tau}} e^{-\frac{\gamma^2}{2}}dQ
$$

Written more simply, it is:

$$
I_- = e^{\frac{1}{2}\left(c/b-1\right)x+\frac{1}{4}\left(c/b-1\right)^2\tau}N(d_-)
$$

$$
d_- = \frac{x}{\sqrt{2\tau}}+\frac{1}{2}\left(\frac{c}{b}-1\right)\sqrt{2\tau}
$$

Thus,

$$
\begin{aligned}
u(x,\tau) &= e^{-\frac{1}{2}\left(\frac{c}{b}-1\right)x+\left(\frac{d}{b}-\frac{1}{4}\left(\frac{c}{b}-1\right)^2\right)\tau}v(x,\tau) \\
&= e^{-\frac{1}{2}\left(\frac{c}{b}-1\right)x+\left(\frac{d}{b}-\frac{1}{4}\left(\frac{c}{b}-1\right)^2\right)\tau}\left[I_+-I_-\right] \\
&= e^{x+\left(\frac{c}{b}+\frac{d}{b}\right)\tau}N(d_+)-e^{\frac{d}{b}\tau}N(d_-)
\end{aligned}
$$

But, $\tau = b(T-t)$ and $x = \ln\left(S/K \right)$ so:

$$
\begin{aligned}
u(x,\tau) &= e^{\ln\left(\frac{S}{K}\right)+\left(\frac{c}{b}+\frac{d}{b}\right)b(T-t)} N(d_+) - e^{\frac{d}{b}b(T-t)} N(d_-) \\
&= \left(\frac{S}{K}\right) e^{(c+d)(T-t)} N(d_+) - e^{d(T-t)} N(d_-)
\end{aligned}
$$

Likewise, $C(S,t) = Ku(x,\tau) \Rightarrow C(S,t) = Se^{(c+d)(T-t)} N(d_+) - Ke^{d(T-t)} N(d_-)$. Finally,

we make the resubstitutions for d_+ and d_-:

$$
\begin{aligned}
d_+ &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}\left(\frac{c}{b}+1\right)\sqrt{2\tau} \\
&= \frac{\ln\left(\frac{S}{K}\right)}{\sqrt{2b(T-t)}} + \frac{1}{2}\left(\frac{c}{b}+1\right)\sqrt{2b(T-t)} \\
&= \frac{\ln\left(\frac{S}{K}\right) + \frac{1}{2}\left(\frac{c}{b}+1\right)(2b)(T-t)}{\sqrt{2b(T-t)}} \\
& \frac{\ln\left(\frac{S}{K}\right) + (c+b)(T-t)}{\sqrt{2b(T-t)}}
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
d_- &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}\left(\frac{c}{b}-1\right)\sqrt{2\tau} \\
&= \frac{\ln\left(\frac{S}{K}\right)}{\sqrt{2b(T-t)}} + \frac{1}{2}\left(\frac{c}{b}-1\right)\sqrt{2b(T-t)} \\
&= \frac{\ln\left(\frac{S}{K}\right) + \frac{1}{2}\left(\frac{c}{b}-1\right)(2b)(T-t)}{\sqrt{2b(T-t)}} \\
& \frac{\ln\left(\frac{S}{K}\right) + (c-b)(T-t)}{\sqrt{2b(T-t)}}
\end{aligned}
$$

Appendix III

Proposition: The differential equation:

$$\frac{\partial C}{\partial t} + bS^2 \frac{\partial^2 C}{\partial S^2} + cS \frac{\partial C}{\partial S} + dC = 0$$

$$C(0,t) = e^{d(T-t)}$$

$$C(S,t) \sim S \; as \; S \to \infty$$

$$C(S,T) = \max(S,1)$$

has solution:

$$C = e^{d(T-t)} + Se^{(c+d)(T-t)} N(d_+) - e^{d(T-t)} N(d_-)$$

$$d_+ = \frac{\ln(S) + (c+b)(T-t)}{\sqrt{2b(T-t)}}$$

$$d_- = \frac{\ln(S) + (c-b)(T-t)}{\sqrt{2b(T-t)}}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{\left(-\frac{1}{2}\right)^2} ds$$

Proof: The proof is very similar to the above proof. We make the three substitutions:

- $\tau = b(T-t)$ which implies that $t = T - \tau/b$, $\frac{d\tau}{dt} = -b$ and $\tau = 0 \Leftrightarrow t = T$
- $S = e^x$ which implies that $x = \ln(S)$
- $C(S,t) = u(x,\tau)$

The final substitution in tandem with the fact that:

$$C(S,T) = \max(S,1) = \max(e^x,1) = 1 + \max(e^x - 1,0)$$

implies that $u(x,0) = 1 + \max(e^x - 1,0)$. The above substitutions yield the differential equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + \left(\frac{c}{b} - 1\right)\frac{\partial u}{\partial x} + \frac{d}{b} u$$

As before, we set

$$u(x,\tau) = e^{\alpha x+\beta\tau} v(x,\tau) = e^{-\frac{1}{2}\left(\frac{c}{b}-1\right)x+\left(\frac{d}{b}-\frac{1}{4}\left(\frac{c}{b}-1\right)^2\right)\tau} v(x,\tau)$$

where v is a function satisfying

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2}$$

Thus,

$$u(x,0) = \max\left(e^x - 1, 0\right) = e^{-\frac{1}{2}\left(\frac{c}{b}-1\right)x} v(x,0)$$

$$\Rightarrow v(x,0) = e^{\frac{1}{2}\left(c/b-1\right)x} + \max\left(e^{\frac{1}{2}\left(c/b+1\right)x} - e^{\frac{1}{2}\left(c/b-1\right)x}, 0\right)$$

Now, if we think of v as two functions:

$$v(x,t) = v_1(x,t) + v_2(x,t)$$

$$v_1(x,0) = e^{\frac{1}{2}\left(c/b-1\right)x}$$

$$v_2(x,0) = \max\left(e^{\frac{1}{2}\left(c/b+1\right)x} - e^{\frac{1}{2}\left(c/b-1\right)x}, 0\right)$$

Now, it is clear that v_2 is the differential equation of the last proof which has already been solved. Hence, we must only solve v_1.

Its solution is:

$$v_1(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4\tau}} e^{\frac{1}{2}\left(c/b-1\right)y} dy$$

Now, using the same change of variable from Appendix II, $Q = \frac{y-x}{\sqrt{2\tau}}$, our integral becomes:

$$v(x,\tau)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-Q^2/2}\varphi(\sqrt{2\tau}Q+x)dQ$$

$$v_1(x,\tau)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\frac{-Q^2}{2}}e^{\frac{1}{2}\left(c/b-1\right)\left(\sqrt{2\tau}Q+x\right)}dQ$$

Notice, this is exactly the same integral as I_- of Appendix II except that it goes from minus infinity to infinity. Hence its solution is:

$$v_1(x,\tau)=e^{\frac{1}{2}\left(c/b-1\right)x+\frac{1}{4}\left(c/b-1\right)^2\tau}N(\infty)=e^{\frac{1}{2}\left(c/b-1\right)x+\frac{1}{4}\left(c/b-1\right)^2\tau}$$

Hence, the solution of the differential equation with the new boundary condition is

$C=u_1+u_2$ where $u_i(x,t)=e^{-\frac{1}{2}\left(\frac{c}{b}-1\right)x+\left(\frac{d}{b}-\frac{1}{4}\left(\frac{c}{b}-1\right)^2\right)\tau}v_i(x,t)$. Thus, u_2 is as in the last proof and $u_1(x,t)=e^{-\frac{1}{2}\left(\frac{c}{b}-1\right)x+\left(\frac{d}{b}-\frac{1}{4}\left(\frac{c}{b}-1\right)^2\right)\tau}v_1(x,t)=e^{\frac{d}{b}\tau}$. Hence, the solution is:

$$C=e^{d(T-t)}+Se^{(c+d)(T-t)}N(d_+)-e^{d(T-t)}N(d_-)$$

$$d_+=\frac{\ln(S)+(c+b)(T-t)}{\sqrt{2b(T-t)}}$$

$$d_-=\frac{\ln(S)+(c-b)(T-t)}{\sqrt{2b(T-t)}}$$

Ad Maiorem Dei Gloriam!

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i If S^2 pays a stock dividend, r is this number; if S2 pays a dollar dividend, r is this dollar dividend divided by the price of S^2

ii See Appendix II for solution

iii This arbitrage argument presented is for the case where S^2 pays no dividends (i.e., $r = 0$).

[iv] Using the notation of the appendices, the differential equations approach involves understanding that C(S,T) = min(S,1) = S - max(S-1,0) = C_1 - C_2 where C_1= S and C_2 = max(S-1,0). Now, C_2 is the boundary condition for the option from Sections I-II solved in Appendix II. Furthermore, C_1 = S solves the requisite differential equation:

$$\frac{\partial C_1}{\partial t} + rS\frac{\partial C_1}{\partial S} + \frac{1}{2}S^2(\sigma_1{}^2 + \sigma_2{}^2 - 2Cov)\frac{\partial^2 C_1}{\partial S^2} - rC_1 = 0$$

Hence, this proves that the formula $C_t^\$ = S_t^{1,\$} - O_t^\$$ where $O_t^\$$ is the option of Sections II-III which pays off $\max(S_T^{1,\$} - S_T^{2,\$}, 0)$ must be correct.

[v] This is because if *v* satisfies $\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$, $u = e^{\alpha x + \beta t} v$, $\alpha = -\frac{a}{2}$, and $\beta = b - \frac{a^2}{4}$, then u satisfies $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + bu$